

# Homology and the stability problem in the Thompson group family

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We prove that Thompson's group  $V$  is acyclic. The strategy of our proof stems from the context of homological stability and stable homology. We first use algebraic K-theory methods to compute the stable homology for automorphism groups of algebraic theories in general, and for the Higman-Thompson groups in particular. This also leads us to the first example where homological stability fails for automorphism groups of algebraic theories. We then provide a general homotopy decomposition technique for classifying spaces originating in McDuff's and Segal's work related to foliations to give a useable characterization of acyclicity. Finally, we apply this to deal with the specific features of Thompson's group.

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# 1 Introduction

Thompson's group  $V$  and its subgroups  $F$  and  $T$  are famous examples of infinite discrete groups. They have attracted much attention ever since they were introduced. One of the reasons why Thompson's groups have been studied so much is that they appear in various guises throughout mathematics. Invented around 1965 in the context of mathematical logic and (un)solvable word problems (see [MT73] and [Tho80]), they have found interpretations that range from universal algebra, category theory, homotopy and shape theory (see, for example, [HH82], [BG84], [FH93], and [FL10]), to the geometry of piecewise-linear maps, Teichmüller theory, mapping class groups, and braids (see the expositions in [FKS12] and [Pen12]), and applications to data structures and search trees (see [STT88] and [Deh10], for instance).

The homology of the subgroup  $F$  has been computed by Brown and Geoghegan [BG84], see also [Bro06], and the homology of the subgroup  $T$  has been computed by Ghys and Sergiescu [GS87]. Yet, Thompson's largest group  $V$  contains many other interesting subgroups whose homologies have not been computed. For example, while the subgroup  $F$  is torsion-free, every finite group can be embedded into Thompson's group  $V$ . And, in particular, the homology of Thompson's group  $V$  itself has been unknown so far. We offer a solution to this long-standing problem here.

Thompson has shown that  $V$  is simple (in the algebraic sense), hence perfect. Furthermore, it is finitely presentable, and of type  $FP_\infty$ , which means that the trivial  $\mathbb{Z}V$ -module  $\mathbb{Z}$  admits a free resolution which is finitely generated in all dimensions, see [Bro87]. It follows that Thompson's group  $V$  belongs to the class of groups where the homology is finitely generated in each degree. We prove here that the homology actually vanishes in positive degrees.

**Theorem 1.1.** *Thompson's group  $V$  is acyclic.*

It has been shown by Brown [Bro92] that the rational homology of  $V$  vanishes. Another proof of this fact has been given in [Far05] in the context of picture groups, also known as braided diagram groups. This was of course not the only evidence that led Brown in [Bro92] to suggest that the statement made in Theorem 1.1 might be true. He also claimed that  $H_j(V) = 0$  for  $j = 1, 2, 3$ , but only the case  $j = 1$  is clear from the (algebraic) simplicity of the group. See [Kap02] for a tour de force proof of the case  $j = 2$ . All of these results are also consequences of Theorem 1.1.

The proof of Theorem 1.1 that we give here involves ideas that belong to homological stability and stable homology. Recall that a diagram

$$G_0 \longrightarrow G_1 \longrightarrow G_2 \longrightarrow G_3 \longrightarrow \cdots \quad (1.1)$$

of (discrete) groups and homomorphisms is called *homologically stable* if there is an unbounded and non-decreasing function  $\lambda$  such that the homomorphisms in the diagram (1.1) induce isomorphisms  $H_j(G_r) \rightarrow H_j(G_{r+1})$  in homology for all  $j$  and  $r$  such that  $r \geq \lambda(j)$ . Regardless if that is the case or not, the homology of the colimit  $G_\infty$  of the diagram (1.1) is called its *stable homology*. Establishing homological stability of a diagram and describing its stable homology typically involve rather different techniques.

In the context of low-dimensional manifolds, homological stability has first been proven by Harer [Har85], and the corresponding stable homology was computed by Madsen and Weiss [MW07], confirming Mumford's famous conjecture. See [Wah13a], [Wah13b], and [Gal13] for state-of-the-art surveys. In this paper, we not only start with (more and less well-known) algebraic descriptions of Thompson's groups and their relatives, but also our methods are essentially algebraic: The results do not rely on nice actions of the groups on nice spaces as many of the previous homological results do. Still, they fit well with the recently emerging connections between Thompson's groups, the geometry of mapping class groups of surfaces, and braid groups. See again [FKS12] and [Pen12] for relevant recent surveys of this emerging interaction.

We are here going to study a canonical diagram

$$V_0 \longrightarrow V_1 \longrightarrow V_2 \longrightarrow V_3 \longrightarrow \cdots \quad (1.2)$$

of groups and homomorphisms such that there are isomorphisms  $V_r \cong V$  with Thompson's group  $V$  for all integers  $r \geq 1$ . We prove the following two results.

**Theorem 1.2.** *The stable homology of the diagram (1.2) is trivial.*

In other words, the stable Thompson group  $V_\infty$  is acyclic. The proof of Theorem 1.2 involves methods from algebraic K-theory.

After calculating the stable homology, we can turn towards homological stability:

**Theorem 1.3.** *The diagram (1.2) satisfies homological stability.*

Because of the existence of an isomorphism  $V_r \cong V$  with Thompson's group  $V$  for all integers  $r \geq 1$ , Theorems 1.2 and 1.3 together imply Theorem 1.1. In fact, Theorem 1.1 is equivalent to the logical conjunction of Theorems 1.2 and 1.3. In this paper, we will prove Theorem 1.2 first, and then deduce Theorem 1.1 from it by means of a homotopy decomposition of the classifying space of Thompson's group  $V$ . Theorem 1.3 follows as indicated. In fact, Theorem 1.3 in itself might not seem extraordinarily substantial once the acyclicity of the groups involved is established. But, the logical interaction between the preceding results works in more than one direction, and this eventually leads to the following counterexample to the homological stability problem for automorphism groups.

**Theorem 1.4.** *The canonical diagram*

$$\mathrm{Aut}(C_{3,0}) \longrightarrow \mathrm{Aut}(C_{3,1}) \longrightarrow \mathrm{Aut}(C_{3,2}) \longrightarrow \mathrm{Aut}(C_{3,3}) \longrightarrow \cdots$$

*of Higman-Thompson groups  $\mathrm{Aut}(C_{3,r})$  does not satisfy homological stability.*

As is well-known, and will be explained in Section 3.4, the Higman-Thompson groups  $\mathrm{Aut}(C_{n,r})$  appear as automorphism groups in universal algebra. (The nota-

tion  $\text{Aut}(C_{n,r}) = V_{n,r}$  is also in use, and in this notation  $V_{2,r} = V_r$  recovers Thompson's group  $V$ .) Theorem 1.4 yields the first known example of an algebraic theory (in the sense of Lawvere [Law63]) where the automorphism groups fail to display homological stability.

Other groups that are related to Thompson's group  $V$ , such as the ones that have been introduced independently by Brin (see [Bri04], [Bri06], and [Bri07]) and Dehornoy (see [Deh05] and [Deh06]), are briefly mentioned in Remarks 3.12 and 5.10.

Because, as often, the methods of the paper may be just as interesting as the result itself, they are presented here in what seems to be an adequate level of generality. In the next Section 2, we explain that the stable homology for diagrams of automorphism groups in any algebraic theory can be expressed in terms of algebraic K-theory. Section 3 provides the necessary background on Thompson's group and Higman-Thompson groups to deduce Theorem 1.2 from that, and it also contains the proof of Theorem 1.4. In Section 4 we present a general homotopy decomposition technique for classifying spaces that goes back to McDuff's and Segal's study of classifying spaces related to foliations [Seg78]. This leads to a usable characterization of acyclicity in Proposition 4.3. The final Section 5 shows how this can be applied to Thompson's group  $V$  in order to deduce Theorem 1.1.

## 2 The stable homology of algebraic theories

In this section we give one definition of the K-theory space  $K(\mathbf{T})$  of an algebraic theory  $\mathbf{T}$ . The model that we have chosen (essentially the plus construction) makes it immediately clear that the (reduced) homology of this space is the (reduced) stable homology of the diagram of automorphism groups of free  $\mathbf{T}$ -algebras. Later models for algebraic K-theory (such as Quillen's categorification of the Grothendieck construction) allow us to manipulate these spaces more easily, and we will use this in the following sections to compute stable homology. Here,

we start with a review of the relevant universal algebra, and end with some examples that demonstrate the interest of the K-theory of algebraic theories beyond the purposes of the present paper.

## 2.1 Algebraic theories

We need to review the basic notions and set up our notations for algebraic theories in the sense of Lawvere [Law63].

We will choose a skeleton **Sets** of the category of finite sets and all maps between them. For each integer  $r \geq 0$  such a category has a unique object, say  $S_r$ , with precisely  $r$  elements, and there are no other objects. For the sake of explicitness, let us choose the model  $S_r = \{a \in \mathbb{Z} \mid 1 \leq a \leq r\}$  for such a set. A set with  $r + s$  elements is the (categorical) sum of a set with  $r$  elements and a set with  $s$  elements.

**Definition 2.1.** An *algebraic theory* is pair consisting of a category  $\mathbf{T}$  together with a functor  $\mathbf{Sets} \rightarrow \mathbf{T}$  that is bijective on objects and that preserves sums.

The image of the set  $S_r$  under the functor  $\mathbf{Sets} \rightarrow \mathbf{T}$  will be written  $T_r$ , so that the object  $T_r$  is the sum in the category  $\mathbf{T}$  of  $r$  copies of the object  $T_1$ .

**Remark 2.2.** Some authors prefer to work with the opposite category  $\mathbf{T}^{\text{op}}$ , so that the object  $T_r$  is the product of  $r$  copies of the object  $T_1$ . For example, this was Lawvere's convention when he introduced this notion in [Law63]. Our convention reflects the point of view that the object  $T_r$  should be thought of as the free  $\mathbf{T}$ -algebra on  $r$  generators, covariantly in  $r$  (or rather in  $\mathbf{Sets}$ ). To make this precise, recall the definition of an algebra for a theory  $\mathbf{T}$ .

**Definition 2.3.** Let  $\mathbf{T}$  be an algebraic theory. A  $\mathbf{T}$ -*algebra* is a presheaf (of sets) on  $\mathbf{T}$  that sends (categorical) sums in  $\mathbf{T}$  to (categorical, i.e. Cartesian) products of sets.

The values of a  $\mathbf{T}$ -algebra are determined up to isomorphism by the value at  $T_1$ , and we will often use the same notation for an algebra and that set.

The Yoneda embedding sends the object  $T_r$  of  $\mathbf{T}$  to a presheaf that is readily checked to be an algebra. Since the Yoneda embedding is fully faithful, we may write  $\mathbf{T}(X, Y)$  to denote the set of morphisms  $X \rightarrow Y$  between  $\mathbf{T}$ -algebras. The definitions unravel to give a natural bijection

$$\mathbf{T}(T_r, X) \cong X^r$$

for  $\mathbf{T}$ -algebras  $X$ , so that  $T_r$  is indeed a free  $\mathbf{T}$ -algebra on  $r$  generators.

An algebraic theory  $\mathbf{T}$ , as a category, becomes symmetric monoidal with respect to the (categorical) sum, and the unit object  $T_0$  for this structure is also an initial object in the category  $\mathbf{T}$ .

**Example 2.4.** It might be helpful to note that algebraic theories can be thought of as generalizations of rings. Every ring  $A$  yields a theory  $\mathbf{Mod}_A$  of  $A$ -modules, where the sets of morphisms  $r \rightarrow s$  are given by the sets  $\mathrm{Hom}_A(A^r, A^s)$  of morphisms of finitely generated free  $A$ -modules. The category of  $\mathbf{Mod}_A$ -algebras is equivalent to the category of  $A$ -modules. Note, however, that already rings can be very complicated in general. This applies even more so for algebraic theories.

We will see other examples at the end of this section.

## 2.2 Automorphism groups and stabilization

Let  $\mathbf{T}$  be any algebraic theory. The automorphisms groups of the free algebras  $T_r$  are our main object of study. We will use the notation  $G_r(\mathbf{T})$  for them.

Given integers  $r, s \geq 0$ , there is a stabilization homomorphism

$$G_r(\mathbf{T}) \longrightarrow G_{r+s}(\mathbf{T}) \tag{2.1}$$

that ‘adds’ the identity of the object  $T_s$  in the sense of the categorical sum  $+$ , and this will also be the notation that we use. More precisely, stabilization sends an

automorphism  $u$  of  $T_r$  to the automorphism of  $T_{r+s}$  that makes the diagram

$$\begin{array}{ccc} T_{r+s} & \dashrightarrow & T_{r+s} \\ \cong \uparrow & & \uparrow \cong \\ T_r + T_s & \xrightarrow{u+T_s} & T_r + T_s \end{array}$$

commute. By abuse of notation, this automorphism of the object  $T_{r+s}$  will sometimes also be denoted by  $u + T_s$ .

**Remark 2.5.** The alert reader will have noticed that we have not specified our choice of isomorphism  $T_r + T_s \cong T_{r+s}$  in the preceding diagram, and we do not need to: All such choices obviously differ by some conjugation, so that they induce the same map in homology, which is all that matters for the purposes of the present text.

Stabilization leads to a diagram

$$G_0(\mathbf{T}) \longrightarrow G_1(\mathbf{T}) \longrightarrow G_2(\mathbf{T}) \longrightarrow G_3(\mathbf{T}) \longrightarrow \cdots \quad (2.2)$$

of groups for every algebraic theory  $\mathbf{T}$ . We will write

$$G_\infty(\mathbf{T}) = \operatorname{colim}_{r \rightarrow \infty} G_r(\mathbf{T})$$

for the colimit of the diagram (2.2).

**Definition 2.6.** The colimit  $G_\infty(\mathbf{T})$  of the groups  $G_r(\mathbf{T})$  with respect to the stabilization maps is the *stable automorphism group* of the algebraic theory  $\mathbf{T}$ .

**Remark 2.7.** In contrast to what the notation might suggest, this need not be the automorphism group of a free  $\mathbf{T}$ -algebra on infinitely many generators.

**Example 2.8.** If  $\mathbf{T} = \mathbf{Sets}$  is the theory of sets, then  $G_\infty(\mathbf{Sets})$  is isomorphic to the group of permutations of an infinite set that move only a finite number of its elements, compare Example 2.14.



**Proposition 2.9.** *For every algebraic theory  $\mathbf{T}$ , the stabilization maps*

$$G_r(\mathbf{T}) \longrightarrow G_{r+1}(\mathbf{T})$$

*are injective.*

*Proof.* It is enough to show that the kernels are trivial. This is clear for  $r = 0$ , since  $T_0$  is initial, so that  $G_0(\mathbf{T})$  is the trivial group. For positive  $r$  we can choose a retraction  $\rho$  of the stabilization map  $\sigma: T_r \rightarrow T_{r+1}$ . If  $u$  is in the kernel, then the diagram

$$\begin{array}{ccc} T_r & \xrightarrow{u} & T_r \\ \sigma \downarrow & & \downarrow \sigma \\ T_{r+1} & \xrightarrow{\text{id}} T_{r+1} \xrightarrow{\rho} & T_r \end{array}$$

commutes and implies  $u = \text{id}$ . □

Let us record the following group theoretical property of the stable automorphism groups. This is presumably well-known already in more or less generality. We will nevertheless include an argument here for completeness' sake.

**Proposition 2.10.** *For every algebraic theory  $\mathbf{T}$ , the commutator subgroup of the stable automorphism group  $G_\infty(\mathbf{T})$  is perfect.*

*Proof.* Given a commutator in the group  $G_\infty(\mathbf{T})$ , we can represent it as  $[u, v]$  for a pair  $u, v$  of automorphisms in the group  $G_r(\mathbf{T})$  for some  $r$ . Allowing us thrice the space, in the group  $G_{3r}(\mathbf{T})$  we have the identity

$$[u, v] + \text{id}_{T_{2r}} = [u + u^{-1} + \text{id}_{T_r}, v + \text{id}_{T_r} + v^{-1}].$$

It therefore suffices to prove that each element of the form  $w + w^{-1}$  is a commutator. This is a version of Whitehead's lemma that holds in every symmetric monoidal category: Whenever there are automorphisms  $w_1, \dots, w_n$  of an object such that their composition  $w_1 \dots w_n$  is the identity, then  $w_1 + \dots + w_n$  is a commutator. We apply this to the category  $\mathbf{T}$  with respect to the monoidal product given by categorical sum  $+$ . □

## 2.3 Algebraic K-theory

We can define the K-theory of an algebraic theory  $\mathbf{T}$  as a generalization of the algebraic K-theory of a ring (or, more precisely, of the category of finitely generated free modules.) We will start with the Grothendieck group  $K_0(\mathbf{T})$ .

The Grothendieck group  $K_0(\mathbf{T})$  of an algebraic theory  $\mathbf{T}$  is the group completion of the abelian monoid of isomorphism classes  $[T_r]$  of objects of  $\mathbf{T}$ , subject to the relations  $[T_r + T_s] = [T_r] + [T_s]$  for all  $r, s$  (and  $[T_r] = [T_s]$  whenever  $T_r \cong T_s$ , because these are isomorphism classes). It follows that  $K_0(\mathbf{T})$  is cyclic, generated by  $[T_1]$ .

For higher algebraic K-theory, there are by now many constructions. For the present purposes, the historically first one, the plus construction [Qui71] (see also [Wag72] and [Lod76]) is the obvious choice. That this produces an interesting theory for rings is widely known, and the extension to general algebraic theories is straightforward in the presence of Proposition 2.10.

The plus construction can be applied to spaces  $X$  for which the fundamental groups have perfect commutator subgroups. It produces a map  $X \rightarrow X^+$  into another connected space  $X^+$  with the same integral homology, and such that the induced maps on fundamental groups are the abelianization. In fact, these two properties characterize the plus construction. By Proposition 2.10, the commutator subgroup of  $G_\infty(\mathbf{T})$  is perfect. Therefore, the plus construction can be applied on  $BG_\infty(\mathbf{T})$  in order to produce another space  $BG_\infty(\mathbf{T})^+$ .

**Definition 2.11.** We will refer to the product

$$K(\mathbf{T}) = K_0(\mathbf{T}) \times BG_\infty(\mathbf{T})^+ \quad (2.3)$$

as the *K-theory space* of the algebraic theory  $\mathbf{T}$ .

**Example 2.12.** For rings  $A$  and their associated theories of  $A$ -modules  $\mathbf{Mod}_A$  as in Example 2.4, this is the ordinary algebraic K-theory of the category of finitely generated free  $A$ -modules.

In general, there seems to be no reason to believe that such an artificial product as in (2.3) would form a meaningful unit. The present case is special for the fact that  $K_0(\mathbf{T})$  is generated by the isomorphism class of the free  $\mathbf{T}$ -algebra  $T_1$  of rank 1. Other constructions of the same homotopy type do not separate the group  $K_0(\mathbf{T})$  of components from the rest of the space. One way or another, note that all components of the algebraic K-theory space  $K(\mathbf{T})$  are equivalent; the group  $K_0(\mathbf{T})$  of components acts transitively on the infinite loop space  $K(\mathbf{T})$ .

## 2.4 Stable homology

Since the plus construction does not change homology, the definition of the algebraic K-theory space immediately gives the following result.

**Proposition 2.13.** *For every algebraic theory  $\mathbf{T}$ , there is an isomorphism*

$$H_*(G_\infty(\mathbf{T})) \cong H_*(K(\mathbf{T}))_{K_0(\mathbf{T})}$$

*between the stable homology of the automorphism groups of the theory  $\mathbf{T}$  and the coinvariants of the homology of the algebraic K-theory space  $K(\mathbf{T})$  under the action of the group  $K_0(\mathbf{T})$  of components.*

*Proof.* Both groups are isomorphic to the homology  $H_*(BG_\infty(\mathbf{T})^+)$  of the plus construction.  $\square$

Ideally, the algebraic K-theory space  $K(\mathbf{T})$  is more accessible and easier to understand than the stable automorphism group  $G_\infty(\mathbf{T})$ . This is not at all clear from its definition. Only the now classical methods of algebraic K-theory developed over half a century allow us to say so, and Proposition 2.13 should be thought of as a computation of the group homology. The following examples give a taste of the flavor of some non-trivial cases.

**Example 2.14.** Consider the initial theory **Sets** of sets. Then  $G_r(\mathbf{Sets}) = \Sigma_r$  is the symmetric group on  $r$  symbols. The algebraic K-theory space  $K(\mathbf{Sets}) = \mathbb{Z} \times B\Sigma_\infty^+$

has been shown to be the infinite loop space underlying the sphere spectrum by Barratt-Priddy, Quillen, and Segal.

**Example 2.15.** Consider the theory **Abel** = **Mod** $_{\mathbb{Z}}$  of abelian groups. Then the automorphism groups  $G_r(\mathbf{Abel}) = \mathrm{GL}_r(\mathbb{Z})$  of the free abelian groups  $\mathbb{Z}^r$  are general linear groups over the ring of integers. By definition, the algebraic K-theory space  $K(\mathbf{Abel})$  is the algebraic K-theory space of the ring of integers.

**Example 2.16.** Consider the theory **Groups** of (all) groups. In this case, the groups  $G_r(\mathbf{Groups}) = \mathrm{Aut}(F_r)$  are the automorphism groups of the free groups  $F_r$  on  $r$  generators. The algebraic K-theory space  $K(\mathbf{Groups})$  has been shown to be the infinite loop space underlying the sphere spectrum by Galatius [Gal11].

In the next section we will proceed to explain how Thompson’s group  $V$  fits in here.

### 3 Thompson’s group and stable homology

In this section we collect the necessary information about Thompson’s group  $V$  and some of its relatives. The approach taken here is based on universal algebra. See [Bro87], [Bro92], and [Bro06], as well as some of the original sources such as [JT61], [MT73], [Hig74], and [Tho80]. For more geometrically minded introductions to Thompson’s groups, see [CF11] and [CFP96], especially Section 6 for the group  $V$ . The latter contains some details from Thompson’s unpublished notes.

#### 3.1 Cantor algebras

Thompson’s group  $V$  arises as the automorphism group of a free algebra in the sense of universal algebra. We start by recalling the underlying algebraic theory **Cantor** of Cantor algebras.

**Definition 3.1.** A *Cantor algebra* is a set  $X$  together with a multiplication

$$\mu : X \times X \longrightarrow X, (x, y) \longmapsto \mu(x, y) = x \cdot y$$

that is a bijection.

The bijection  $X \rightarrow X \times X$  inverse to  $\mu$  will be written  $(\lambda, \rho)$  in components, so that we can express the fact that these maps are inverse to each other as equations

$$\lambda(x) \cdot \rho(x) = x \tag{3.1}$$

as well as  $\lambda(x \cdot y) = x$  and  $\rho(x \cdot y) = y$ .

By induction, for any given  $r \geq 1$ , it is possible to choose a bijection

$$X^r \cong X \tag{3.2}$$

that is natural in Cantor algebras  $X$ . However, such a natural bijection is not canonical. In fact, any choice of bracketing a product of  $r$  factors can be used, and permutations produce even more examples. In a sense, Thompson's group  $V$  has been invented to classify the possibilities.

**Examples 3.2.** The empty set and all singletons are Cantor algebras in a unique way, and these are clearly the only finite Cantor algebras. Cantor's pairing function  $\mu(x, y) = x + (x + y)(x + y + 1)/2$  makes the set of natural numbers a Cantor algebra.

The only examples of Cantor algebras of interest to us are the free ones, to be introduced now: For each integer  $r \geq 1$  there exists a free Cantor algebra  $C_r$  with the property that there is a bijection

$$\mathbf{Cantor}(C_r, X) \cong X^r \tag{3.3}$$

natural in Cantor algebras  $X$ , see [Hig74].

For example, we have  $C_0 = \emptyset$ . For abstract reasons, there are canonical identifications

$$C_{r+s} \cong C_r + C_s, \tag{3.4}$$

where  $+$  is the categorical sum in the category of Cantor algebras.

Natural isomorphism  $X^r \cong X$  as in (3.2) combine to give natural bijections

$$\mathbf{Cantor}(C_r, X) \cong X^r \cong X \cong \mathbf{Cantor}(C_1, X),$$

so that Yoneda's lemma implies the existence of an isomorphism

$$C_r \cong C_1. \quad (3.5)$$

of Cantor algebras for all  $r \geq 1$ . Note, however, that such an isomorphism depends on the choice of the natural isomorphism  $X^r \cong X$ , hence is not canonical, as explained above.

It is also easy to chase these definitions in order to obtain an explicit formula for an isomorphism

$$C_2 \cong C_1. \quad (3.6)$$

To do so, we will use  $(x_1, \dots, x_r)$  for the sequence of elements in  $C_r$  that corresponds to the identity under the bijection (3.3). (The ambient Cantor algebra can always be inferred from the context.) Then, an isomorphism  $C_2 \rightarrow C_1$  is given on generators by  $x_1 \mapsto \lambda(x_1)$  and  $x_2 \mapsto \rho(x_1)$ ; the inverse  $C_1 \rightarrow C_2$  inverse sends the generator  $x_1$  to  $x_1 \cdot x_2$ .

## 3.2 Product structures

While the main interest in Cantor algebras arguably comes from their somewhat exotic behavior under categorical sums, they also admit a well-behaved product structure. This will be a key ingredient when we apply multiplicative structures to determine the algebraic K-theory of the category of finitely generated free Cantor algebras. It will also be used in Remark 3.12 on higher Thompson groups.

If  $X$  and  $Y$  are Cantor algebras with multiplications  $\mu_X$  and  $\mu_Y$ , respectively, then so is the Cartesian product  $X \times Y$  with respect to the multiplication

$$\mu_{X \times Y}(x, y, x', y') = (x \cdot x', y \cdot y').$$

In other words, in contrast to the categorical sum, the categorical product of Cantor algebras is supported on the Cartesian product of the underlying sets. This is a general feature of algebraic theories and should be familiar from the theories of groups and abelian groups. The categorical sum is different from the disjoint union in both of these cases, whereas the product is always Cartesian.

**Remark 3.3.** It is known that the categorical sum and product of  $\mathbf{T}$ -algebras for an algebraic theory  $\mathbf{T}$  need not be compatible in the sense that the canonical morphism

$$(X \times Y) + (X' \times Y) \longrightarrow (X + X') \times Y \quad (3.7)$$

need not be an isomorphism. This failure of compatibility happens, for example, for the theory of groups, the theory of abelian groups (where finite sums are finite products), and the theory of pointed sets. On the other hand, it is known that Cantor algebras form a topos. See, for example, Johnstone ([Joh77] and [Joh85]), who attributes this observation to Freyd. Briefly, a Cantor algebra is a presheaf on the free (cancellative) monoid with two generators  $\lambda$  and  $\rho$  that is a sheaf with respect to the Grothendieck topology generated by the cover that consists of these two generators. In particular, the category of Cantor algebras is distributive in the sense that the canonical morphism (3.7) is a natural isomorphism.

Here is another possible source for confusion: For every Cantor algebra  $X$  there is a given bijection  $X \times X \cong X$ , namely  $\mu = \mu_X$ . However, this bijection will only rarely be a(n iso)morphism of Cantor algebras. As a consequence, the following observation, while easy, is non-trivial.

**Proposition 3.4.** *There are isomorphisms*

$$C_r \times C_s \cong C_{rs} \quad (3.8)$$

*for all non-negative integers  $r$  and  $s$ .*

*Proof.* The isomorphism (3.6) of Cantor algebras  $C_1 + C_1 \cong C_2 \cong C_1$  yields an isomorphism

$$C_r + C_r \cong C_{2r} \cong C_r$$

of Cantor algebras that is natural on the category **Cantor**. As in our argument for the isomorphism (3.5), the Yoneda lemma implies that there exists an isomorphism

$$C_1 \cong C_1 \times C_1 \quad (3.9)$$

of Cantor algebras. The existence of an isomorphism (3.5) together with the fact that  $C_0 = \emptyset$  imply the result.  $\square$

**Remark 3.5.** The isomorphism (3.9) can be made explicit if needed: The generator  $x_1$  of  $C_1$  goes to the pair  $(\lambda(x_1), \rho(x_1))$ . However, this does not mean that the isomorphism is given for all elements by  $(\lambda, \mu)$  in components.

We can use Proposition 3.4 to prove the following result.

**Theorem 3.6.** *The algebraic K-theory space  $K(\mathbf{Cantor})$  is contractible.*

*Proof.* This is a formal consequence of the multiplicative structure on the category **Cantor**: The presence of such a structure implies that  $K(\mathbf{Cantor})$  is the underlying (infinite loop) space of a *ring* spectrum, so that all homotopy groups  $\pi_n K(\mathbf{Cantor})$  are modules over the ring  $\pi_0 K(\mathbf{Cantor}) = K_0(\mathbf{Cantor})$ . But, that ring is trivial: The isomorphism  $C_2 \cong C_1$  of Cantor algebras translates into the relation  $2 = 1$  in that ring, so that  $1 = 0$  holds. Consequently, all homotopy groups of  $K(\mathbf{Cantor})$  are trivial, and that proves the theorem.

It remains to explain why  $K(\mathbf{Cantor})$  indeed is the underlying (infinite loop) space of a ring spectrum. In general, just as any symmetric monoidal category gives rise to a(n algebraic K-theory) spectrum, a compatible multiplication on the category induces a multiplication on that spectrum. For example, it suffices to have a *bimonoidal category* (or *rig category*). These are symmetric monoidal categories with respect to a bifunctor  $\oplus$ , that have another monoidal structure  $\otimes$ , and that come with coherent distributivity isomorphisms

$$(X \otimes Y) \oplus (X' \otimes Y) \longrightarrow (X \oplus X') \otimes Y.$$

Often, the sum  $\oplus$  will be the categorical sum  $+$ . (For example, this is the case for modules over a commutative ring with  $\otimes$  the usual tensor product, and also



for pointed sets, with  $\otimes = \wedge$ .) In *distributive categories* it is also the case that the second monoidal structure  $\otimes = \times$  is the categorical product.

In the case at hand, the category of Cantor algebras is distributive, see Remark 3.3, and the category **Cantor** of finitely generated free Cantor algebras is closed under finite sums and finite products by Proposition 3.4. This ensures the existence of the multiplicative structure that was used above.  $\square$

**Remark 3.7.** The more elementary multiplicative structure up to homotopy on the plus-construction model for algebraic K-theory is explained by Loday [Lod76] in the case of a commutative ring  $A$ . In this sense, the finitely generated free Cantor algebras behave like the finitely generated free modules over a commutative ring  $A$  such that  $A^2 \cong A$  as  $A$ -modules, in other words, like modules over the trivial ring.

### 3.3 Thompson's group

We will write

$$V_r = G_r(\mathbf{Cantor})$$

for the automorphism group of the free Cantor algebra  $C_r$  on  $r$  generators. Clearly, this is only interesting when  $r \geq 1$ : The group  $G_0(\mathbf{Cantor})$  is trivial, consisting of the identity of the empty set. The groups  $V_r$  for  $r \geq 1$  are isomorphic to each other (but not canonically). It will be worth the effort to spell out the abstract argument for this, so that these isomorphism can be made reasonably explicit, if needed: Once the choice of an isomorphism  $C_r \cong C_1$  such as (3.5) is made, this results in an isomorphism

$$V_r \cong V_1 \tag{3.10}$$

by conjugation with that choice, and up to conjugation within  $V_1$  (or  $V_r$ ), this is independent of the choice of the isomorphism  $C_r \cong C_1$ .

The groups  $V_r$  for  $r \geq 1$  are also isomorphic to Thompson's group  $V$ . For the purposes of the present paper, this will be our definition of the (isomorphism type) Thompson's group  $V$ .

We are now ready to prove Theorem 1.2 that says that the homology of the stable group  $V_\infty$  is trivial.

*Proof of Theorem 1.2.* By definition of the stable group  $V_\infty$ , we have

$$V_\infty = G_\infty(\mathbf{Cantor}),$$

where **Cantor** is the theory of Cantor algebras. By Proposition 2.13, the stable homology vanishes if the algebraic K-theory space  $K(\mathbf{Cantor})$  is contractible, and that is the case by Theorem 3.6.  $\square$

### 3.4 Higman-Thompson groups

There are generalizations  $V_{n,r}$  for  $n \geq 2$  of Thompson's group that have been studied by Higman [Hig74] and Smirnov [Smi75]. These groups are usually called *Higman-Thompson groups*. They are defined as the automorphism groups of the free algebras  $C_{n,r}$  on  $r$  generators for the theory **Cantor** <sub>$n$</sub>  where the algebras are the sets  $X$  together with a bijection  $X^n \rightarrow X$ . In our previous notation, we can write **Cantor** = **Cantor**<sub>2</sub>, and  $V_r = V_{2,r}$  so that Thompson's group  $V$  arises in the case  $n = 2$ .

Most of the results about Cantor algebras and their products from Section 3.2 generalize immediately to the theories **Cantor** <sub>$n$</sub>  for  $n \geq 2$ . In particular, the category of **Cantor** <sub>$n$</sub> -algebras is a topos, and as such the categorical sum and product turn it into a distributive category as in Remark 3.3.

The one exception is the generalization of Proposition 3.4 that we need in order to obtain a ring structure on the algebraic K-theory. The statement generalizes, but the proof needs an extra argument:

**Proposition 3.8.** *For all  $n \geq 2$ , the products  $C_{n,r} \times C_{n,s}$  are finitely generated and free.*

*Proof.* As in the proof of Proposition 3.4, distributivity allows us to reduce to the case  $r = 1 = s$ . The argument given there generalizes to show that

$$(C_{n,1})^n \cong C_{n,1} \quad (3.11)$$

as **Cantor**<sub>n</sub>-algebras. This already proved the claim in the case  $n = 2$ . In general:

The projections can be used to define morphisms  $(C_{n,1})^2 \rightarrow (C_{n,1})^{n-2}$  of **Cantor**<sub>n</sub>-algebras. The bare existence of such morphisms implies that the algebra  $(C_{n,1})^2$  is a retract of a finitely generated free **Cantor**<sub>n</sub>-algebra (3.11). This shows that  $(C_{n,1})^2$  is finitely generated. Furthermore, it also implies that  $(C_{n,1})^2$  is a subalgebra of the free algebra  $C_{n,1}$ , and therefore it is free itself [Hig74].  $\square$

**Remark 3.9.** The stable groups  $V_{n,\infty}$  and their variants that belong to generalizations of Thompson's subgroups F and T have already appeared in [Bro87] and again in [Bri07], for example, but not in the contexts of homology and stability.

### 3.5 A counterexample

The following result is a restatement of Theorem 1.4 from the introduction.

**Theorem 3.10.** *The automorphism groups in the algebraic theory **Cantor**<sub>3</sub> do not satisfy homological stability.*

*Proof.* As in the proof of Theorem 3.6, we start by identifying the Grothendieck ring. The existence of an isomorphism  $C_{3,1} \cong C_{3,3}$  between free Cantor algebras translates into the relation  $3 = 1$  in the ring  $K_0(\mathbf{Cantor}_3)$ , so that  $2 = 0$  holds. It follows that the Grothendieck ring  $K_0(\mathbf{Cantor}_3)$  is either  $\mathbb{Z}/2$  or trivial. In both bases, the homotopy groups of the K-theory *spectrum* are modules over  $\mathbb{Z}/2$ . Consequently, the second homology group of the K-theory *space*  $K_0(\mathbf{Cantor}_3)$  is killed by 2 as well, and so is the stable homology of the groups  $V_{3,r}$ . In contrast, it is known that  $H_2(V_{3,1}) \cong \mathbb{Z}/4$ , see [Kap02]. Because  $C_{3,1} \cong C_{3,3}$ , it follows that  $H_2(V_{3,r}) \cong \mathbb{Z}/4$  for all odd  $r$ . Homological stability would imply that the

second stable homology were also cyclic of order 4, in contradiction to the above.  $\square$

To my knowledge, this is the first example of an algebraic theory where the automorphism groups fail to display homological stability.

**Remark 3.11.** It might be worth noting that the example is also not representation stable in the sense of Church and Farb [CF13] (see also [Far]). The reason is simple: The symmetric groups act trivially (induced by conjugation) on the homology groups of automorphism groups of algebraic theories, so that the notions of homological stability and representation stability are actually equivalent in the present situation.

**Remark 3.12.** The Higman-Thompson groups have been generalized still further in [Bri04] to *higher dimensional Thompson groups*. These can be described in universal algebraic terms as well, using the tensor product  $\mathbf{T}_1 \otimes \mathbf{T}_2$  of algebraic theories [Law68] that generalizes the tensor product of rings. Its algebras are the  $\mathbf{T}_1$ -algebras in the category of  $\mathbf{T}_2$ -algebras (instead of sets). For example, the algebras for the theory  $\mathbf{Cantor} \otimes \mathbf{Cantor}$  are the Cantor algebras  $\mu: X^2 \rightarrow X$  together with another Cantor algebra structure  $\nu: X^2 \rightarrow X$  such that  $\nu$  is a morphism of Cantor algebras in the usual sense. (Here  $X^2 = X \times X$  is the product Cantor algebra as explained in Section 3.1 above.) For any integer  $d \geq 2$ , we may then consider the algebraic theory  $\mathbf{Cantor}^{\otimes d}$ , the  $d$ -th iterated tensor product of  $\mathbf{Cantor}$  with itself. The automorphism group  $dV$  of the free  $\mathbf{Cantor}^{\otimes d}$ -algebra on one generator is called the *d-dimensional Thompson group*. (The case  $d = 1$  is Thompson's group  $V = 1V$ .) There are also variants

$$dV_{n,r} = G_r(\mathbf{Cantor}_n^{\otimes d})$$

with  $\mathbf{Cantor} = \mathbf{Cantor}_2$  replaced by  $\mathbf{Cantor} = \mathbf{Cantor}_n$  and any number  $r \geq 1$  of generators. See [Bri04] again. And again it seems reasonable to believe that the arguments that have been presented so far apply equally well to these groups. Since this would not significantly increase the substance of our results, this will not be pursued here.

## 4 Segal decompositions of classifying spaces

In this section we will present a homotopy decomposition technique for classifying spaces that generalizes ideas from McDuff's and Segal's work [Seg78] (see also [HM83]) on classifying spaces related to foliations. We then use that in order to give a characterization of acyclicity in Proposition 4.3.

We will be using the standard bar construction model for our classifying spaces: The  $p$ -simplices in the universal (free and contractible)  $G$ -space  $EG$  are the sequences  $(g_0, \dots, g_p)$  of length  $p + 1$  in  $G$ , and the classifying space  $BG$  is the orbit space.

**Definition 4.1.** Let  $G$  be a discrete group. A set  $\mathcal{S}$  of subgroups  $S \leq G$  will be called a *Segal collection* if for every finite family of group elements  $g_j \in G$  and subgroups  $S_j \in \mathcal{S}$  there exists potentially larger subgroups  $S'_j \geq S_j$  in the set  $\mathcal{S}$  and a single group element  $g \in G$  such that we have  $g^{-1}g_j \in S'_j$  for all  $j$ .

**Proposition 4.2.** *Given a Segal collection  $\mathcal{S}$  of subgroups  $S \leq G$  the map*

$$\operatorname{hocolim}_{\mathcal{S}} BS \longrightarrow BG$$

*induced by the inclusions  $BS \subseteq BG$  is an equivalence.*

Since all maps in the  $\mathcal{S}$ -diagram of classifying spaces  $BS$  are inclusions, the homotopy colimit on the left hand side is really the union of the subspaces  $BS$  inside  $BG$ . The invariant formulation seems preferable for conceptual reasons.

*Proof.* For each subgroup  $S$  there is an inclusion  $ES \subseteq EG$ , but the image  $EG$  is not  $G$ -invariant unless we have  $S = G$ . As one remedy of this defect, we can consider the orbit closure  $G \cdot ES$  which has as  $p$ -simplices all sequences  $(g_0, \dots, g_p)$  such that  $g_0^{-1}g_k \in S$  for all  $1 \leq k \leq p$ . Note that the inclusions  $G \cdot ES \subseteq EG$  induce the inclusions  $BS = (G \cdot ES)/G \subseteq EG/G = BG$  upon passage to orbits. The union of all the  $G \cdot ES$ , where  $S$  ranges over the poset  $\mathcal{S}$ , is also a  $G$ -invariant subspace, and the inclusions induce, upon passage to orbits, the map in the statement. Since

the group  $G$  necessarily acts freely on every  $G$ -invariant subspace of  $EG$ , it suffices to show that the union of the  $G \cdot ES$  is contractible in order to prove the proposition. It now suffices to show that for every finite set of simplices  $\sigma_j \subseteq G \cdot ES_j$ , a cone on their union also lies in the union of the  $G \cdot ES$ . Let  $g_j$  denote the 0-th vertex of  $\sigma_j = (g_j, \dots)$ . By hypothesis, we can find subgroups  $S'_j \geq S_j$  in  $\mathcal{S}$  and an element  $g \in G$  such that  $g^{-1}g_j \in S'_j$  for all  $j$ . But this means that the cones  $g \star \sigma_j = (g, g_j, \dots)$  on the simplices  $\sigma_j$  with cone vertex  $g$  are contained in  $G \cdot ES'_j$ , and we are done.  $\square$

Proposition 4.2 determines the homology of the group  $G$  in terms of the homologies of the subgroups  $S$  and the spectral sequence

$$H_p(\mathcal{S}; S \mapsto H_q(S)) \implies H_{p+q}(G). \quad (4.1)$$

But, this fact alone is not enough, for example, to deduce the acyclicity of the group  $G$ , assuming only the acyclicity of the subgroups  $S$  in Segal collection. Some additional information on the homology of the indexing poset  $\mathcal{S}$  is needed, and contractibility is certainly sufficient. This leads to the following characterization of acyclicity.

**Proposition 4.3.** *A group  $G$  is acyclic if and only if it admits a Segal collection  $\mathcal{S}$  of acyclic subgroups  $S \leq G$  such that the poset  $\mathcal{S}$  is contractible.*

*Proof.* We have already explained that the more interesting direction follows from Proposition 4.2 and the spectral sequence (4.1). That this criterion actually characterizes acyclicity is also easy to see: If a group  $G$  is known to be acyclic, then  $\mathcal{S} = \{S\}$  with  $S = G$  is a Segal collection of acyclic subgroups on a contractible poset.  $\square$

We will next see how this characterization of acyclicity can be applied to Thompson's group  $V$ .

## 5 Clones and applications

In this section we will tie up the threads from the previous sections in order to show that Thompson's group  $V$  is acyclic.

For brevity, let us write  $C = C_1$  for the free Cantor algebra on one generator, so that Thompson's group  $V$  is its automorphism group. The following terminology will be useful.

**Definition 5.1.** A *clone* of  $C$  is a non-trivial and proper, finitely generated subalgebra  $X$  of  $C$ .

**Remark 5.2.** There is no need to require that  $X$  is a free factor of  $C$ , because this is automatic here, see [Hig74]. In other words, there is always another clone  $Y$  such that the inclusions induce an isomorphism  $X + Y \cong C$  of Cantor algebras, and there are (non-canonical) isomorphisms  $X \cong C \cong Y$ . (Hence that name.) Conversely, whenever we choose an isomorphism  $C \cong C + C$ , the images of the two summands are clones of  $C$ . This process will also be referred to as *cloning*.

**Proposition 5.3.** *If  $X$  is a clone, then the subgroup  $V(X)$  of Thompson's group  $V$  that consists of the elements that fix all of  $X$  pointwise is isomorphic to Thompson's group  $V$ .*

*Proof.* Let  $c$  be the preferred basis element of  $C$ . Up to isomorphism, we can assume that  $X$  is the clone generated by  $\lambda(c)$ . Then the subgroup  $V(X)$  in question consists of those elements that act on the clone generated by  $\rho(c)$ , and the latter is isomorphic with  $C$ .  $\square$

**Remark 5.4.** We can offer the following geometric interpretation of the preceding result. If we think of  $V$  as a subgroup of the homeomorphism group of the Cantor space  $\{0,1\}^\omega$ , the subgroup  $V(X)$  in question is the subgroup of elements that leave all elements fixed that start with 0. Or, if we think of it as a group of right-continuous bijections of the interval  $[0,1[$ , then the subgroup  $V(X)$  is the subgroup that fixes the subinterval  $[0,1/2[$  pointwise. In both case, the stated

result is easy to ‘see.’ The proof presented above reflects the bias of the present paper, not of its author.

**Proposition 5.5.** *The subgroups  $V(X)$  of Thompson’s group  $V$  form a Segal collection.*

*Proof.* Let us first see that, if  $X$  and  $Y$  are clones, then there are clones  $X' \subseteq X$  and  $Y' \subseteq Y$  such that  $X'$  and  $Y'$  are disjoint: If we already have  $X \cap Y = \emptyset$ , then there is nothing to do. Otherwise  $X \cap Y$  is another clone. We can then clone that to obtain  $X \cap Y = X' + Y'$ , and use the pieces. It follows inductively that finitely many clones can be refined so that they become pairwise disjoint: Make the second one disjoint to the first one as indicated, then make the third one disjoint from the first one and then from the second one, and so on.

Now, given finitely many group elements  $v_j$  and subgroups  $V(X_j)$  for finitely many indices  $j$ , we can use what we have just shown to make the clones  $X_j$  smaller, that is  $X_j \supseteq X'_j$  for some clones  $X'_j$ , so that the smaller ones are pairwise disjoint. This implies that the subgroups of elements which stabilize them pointwise become larger, that is  $V(X_j) \leq V(X'_j)$ . Since we would also like the images  $v_j(X'_j)$  to be disjoint, we may have to apply this argument again to these, to get clones  $Y_j \subseteq v_j(X'_j)$  that are disjoint, and then replace the  $X'_j$  by the  $v_j^{-1}(Y_j)$ . Summing up, we can assume that both the  $X_j$  and also the images  $v_j(X_j)$  are pairwise disjoint. We can also assume (by cloning one more piece) that they do not span  $C$ .

It remains to observe that there is an automorphism  $v \in V$  of  $C$  that agrees with the  $v_j$  on the  $X_j$ . This is so because the complements of the sum of the  $X_j$  and of the sum of the  $v_j(X_j)$  are both (abstractly) isomorphic to  $C$ . The result follows, because  $v^{-1}v_j \in V(X_j)$  if and only if  $v$  and  $v_j$  agree on  $X_j$ .  $\square$

**Remark 5.6.** It may be worth to note that Proposition 5.3 implies that the collection from Proposition 5.5 (together with our characterization of acyclicity in Proposition 4.3) cannot be used to show that Thompson’s group  $V$  is acyclic,



although, it will follow in retrospect that the subgroups in this collection are acyclic.

Let now  $X_\infty = (X_1 > X_2 > X_3 > \dots)$  be a descending sequence of clones in  $C$ . The set of all such clone sequences becomes a poset when we define  $X'_\infty \leq X_\infty$  if and only if  $X'_r \subseteq X_r$  for all  $r$ . If  $X_\infty$  and  $Y_\infty$  are clone sequences such that  $X_1$  and  $Y_1$  are disjoint, then  $X_r$  and  $Y_r$  are disjoint for all  $r$ . Let us use the notation

$$V(X_\infty) = \bigcup_j V(X_j)$$

for the subgroup of elements of Thompson's group  $V$  that fix some  $X_r$  pointwise. Clearly, if the clone sequences satisfy  $X'_\infty \leq X_\infty$ , then the subgroups satisfy the opposite inequality, that is  $V(X'_\infty) \geq V(X_\infty)$ . If  $X_\infty$  and  $Y_\infty$  are clone sequences such that  $X_1$  and  $Y_1$  are disjoint and not complementary, then  $X_r$  and  $Y_r$  are disjoint and not complementary for all  $r$ . In that case, the sums  $X_r + Y_r$  form a clone sequence  $X_\infty + Y_\infty$ , and  $V(X_\infty + Y_\infty) = V(X_\infty) \cap V(Y_\infty)$ .

**Proposition 5.7.** *The set of subgroups  $V(X_\infty)$  of Thompson's group  $V$  is a Segal collection.*

*Proof.* The proof is very similar to the proof of Proposition 5.5: We start again by showing that any two clone sequences  $X_\infty$  and  $Y_\infty$  can be made smaller to be disjoint: If  $X_r$  and  $Y_r$  are disjoint for some  $r$ , then we can just replace the clone sequences by their tails from  $r$  on. Otherwise, we use the clone sequence  $X_\infty \cap Y_\infty$  of the intersections  $X_r \cap Y_r$ , which is smaller than both of them, and clone it to obtain  $X_\infty \cap Y_\infty = X'_\infty + Y'_\infty$ .

The rest of the proof is formally the same of the one of Proposition 5.5, except for one remark: If we choose finitely many group elements  $v$  in finitely many subgroups  $V(X_\infty)$ , then they each leave some  $X_r$  pointwise fixed. While these clones  $X_r$  will depend on the corresponding  $v$ , we can further refine them to ensure that there is one index  $r$  that works for all our group elements  $v$ .  $\square$

**Proposition 5.8.** *The subgroup  $V(X_\infty)$  is isomorphic to the stable group  $V_\infty$ .*

*Proof.* This is a consequence of Proposition 5.3 above: There are isomorphism

$$V_r \xrightarrow{\cong} V(X_r)$$

that are compatible with the stabilization maps and inclusions, respectively. The result follows by passage to colimits.  $\square$

The reader will notice that we are now in a better position than in Remark 5.6: We already know that the subgroups in our collection are acyclic by Theorem 1.2.

**Proposition 5.9.** *The poset of clone sequences is contractible.*

*Proof.* It suffices to show that every finite set  $\mathcal{P}$  of clone sequences is contained in another (finite) set  $\mathcal{Q}$  that is contractible. In order to achieve this, we may choose for every clone sequence  $X_\infty$  in  $\mathcal{P}$  a smaller clone sequence  $X'_\infty \leq X_\infty$  such that the  $X'_\infty$  are pairwise disjoint, and not complementary, so that their (finite) sums are also clone sequences. It follows that the poset  $\mathcal{P}'$  spanned by the  $X'_\infty$  and their (finite) sums is (the subdivision of) a full simplex. Now let  $\mathcal{Q}$  be the union of  $\mathcal{P}$  and  $\mathcal{P}'$ . This obviously contains  $\mathcal{P}$  as a subset, so that it suffices to show that  $\mathcal{Q}$  is contractible. Consider the map  $r$  on  $\mathcal{Q}$  that sends an element  $Y_\infty$  to the largest element of  $\mathcal{P}'$  contained in it. This is a deformation retraction from  $\mathcal{Q}$  to  $\mathcal{P}'$ , since  $r$  is the identity on  $\mathcal{P}'$ , and  $r \leq \text{id}$  (hence  $r \simeq \text{id}$ ) by definition.  $\square$

*Proof of Theorem 1.1.* The acyclicity of Thompson's group  $V$  follows from the preceding two Propositions 5.7 and 5.9 as well as our general characterization of acyclicity in Proposition 4.3.  $\square$

**Remark 5.10.** Several non-trivial extensions

$$1 \longrightarrow N \longrightarrow G \longrightarrow V \longrightarrow 1 \tag{5.1}$$

of Thompson's group  $V$  are known. For example, one such extension  $G$  is the *braided Thompson group* that was discovered independently by Brin (see [Bri06] and [Bri07]) and Dehornoy (see [Deh05] and [Deh06]), and which is an extensions of Thompson's group  $V$  by a colimit of Artin's pure braid groups. In other

examples, the group  $G$  is the universal mapping class group of genus zero, or the asymptotically rigid mapping class group of infinite genus. See [FKS12], where these kinds of extensions are surveyed and put into a unifying context of *cosimplicial symmetric group extensions*. The authors also describe the action of the Grothendieck-Teichmüller group on some completions of the groups involved. Theorem 1.1 implies that the kernel  $N \rightarrow G$  of the extension (5.1) is an integral homology isomorphism as soon as the action of  $V$  on the homology of  $N$  is trivial. And, since  $V$  does not have any proper subgroup of finite index, this might be the case here more often than in general.

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